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Numerical Solution of P.D.E

الصورة العامة لمعادلة تفاضلية من الرتبة الثانية :

$$\boxed{Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu + G = f(x,y)}$$

- **Elliptic**: $B^2 - 4AC < 0$
- **Parabolic**: $B^2 - 4AC = 0$
- **hyperbolic**: $B^2 - 4AC > 0$

لحل مسألة الـ P.D.E نتبع الخطوات التالية

- **Given:** المعادلة التفاضلية والـ I.C والـ B.C.
- **Req:** حل المعادلة والحصول على $u(x,y)$.
- **Sol:** نظرا لصعوبة الحصول على "exact Sol." للمعادلة فإننا نلجأ لإيجاد حلول تقريبية وذلك عن طريق تقسيم منطقة الحل إلى مجموعة من النقاط ونوجد قيمة "u" عند كل نقطة عن طريق التعويض في القانون ~~الحسب~~ نوع المعادلة.

ملحوظة

لإثبات الصورة النهائية لكل نوع من المعادلات نستخدم القوانين

- $u_{x_{i,j}} = \dfrac{u_{i+1,j} - u_{i,j}}{h}$ "or" $u_{x_{i,j}} = \dfrac{u_{i,j} - u_{i-1,j}}{h}$

- $u_{x_{i,j}} = \dfrac{u_{i+1,j} - u_{i-1,j}}{2h}$

- $u_{xx_{i,j}} = \dfrac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$

* نفس القوانين للمشتقات في الـ "y" $\xleftarrow{\text{حيث}}$ التغيير في "j".

arabolic eqn.

لدينا معادلة 1-D heat eqn.

$$\boxed{U_t = \alpha U_{xx}}$$

Sol.

- Explicit Method
- Implicit method
- Crank-Nicolson

1- Explicit Method

يعتمد كل صف في الحساب على الصف الذي يسبقه ولذلك لضمان الحصول على أقل خطأ نختار الـ h,k بحيث

$$0 \leq \mu \leq \frac{1}{2}$$ eqn is stable

ويتم الحصول على الحل من المعادلة الآتية

$$U^{j+1} = A U^{j} + C$$

حيث

$$\begin{bmatrix} U_1^{j+1} \\ U_2^{j+1} \\ \vdots \\ U_{n-1}^{j+1} \end{bmatrix} = \begin{bmatrix} (1-2\mu) & \mu & 0 & \cdots \\ \mu & (1-2\mu) & \mu & \cdots \\ 0 & \mu & (1-2\mu) & \mu \cdots \\ \vdots & & \vdots & \end{bmatrix} \begin{bmatrix} U_1^{j} \\ U_2^{j} \\ \vdots \\ U_{n-1}^{j} \end{bmatrix} + \mu \begin{bmatrix} \alpha_0 \\ 0 \\ 0 \\ \vdots \\ \beta_0 \end{bmatrix}$$

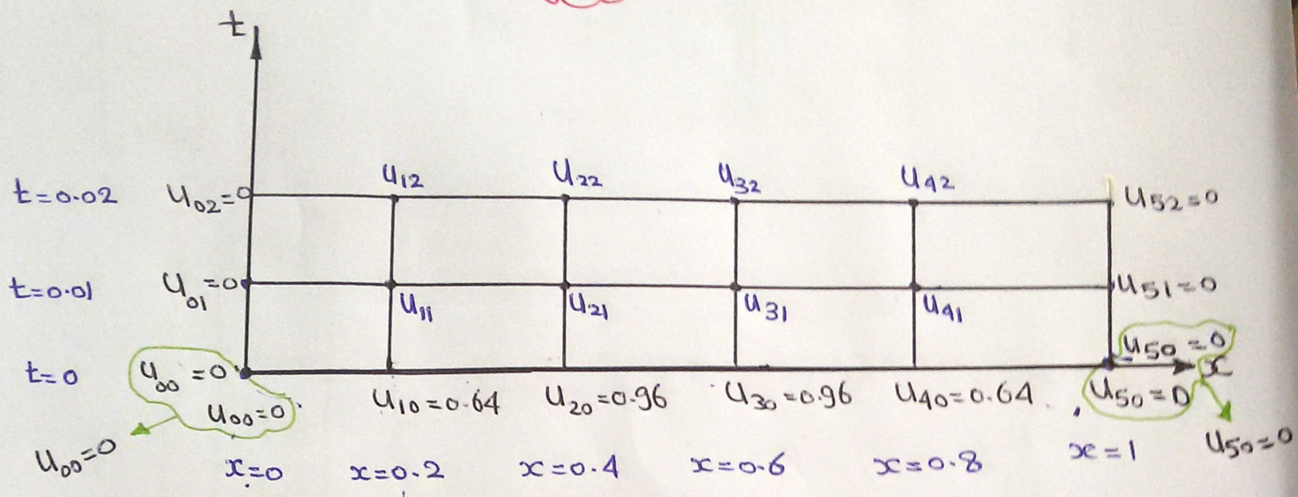
$$\alpha_0 = U(0,t) \quad , \quad \beta_0 = U(L,t) \quad , \quad \mu = \frac{\alpha k}{h^2}$$

proximate the Sol. of $U_t = U_{xx}$, $0 \le x \le 1$, $0 \le t \le 0.02$

$U(0,t) = U(1,t) = 0$

$U(x,0) = 4x - 4x^2$, $h = 0.2$, $k = 0.01$

Sol.



$$\mu = \frac{\alpha k}{h^2} = \frac{0.01}{(0.2)^2} = 0.25$$

So

$$\begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.25 & 0 & 0 \\ 0.25 & 0.5 & 0.25 & 0 \\ 0 & 0.25 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0.64 \\ 0.96 \\ 0.96 \\ 0.64 \end{bmatrix} + 0.25 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$(1-2\mu)$

$$\therefore \boxed{\begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \\ u_{41} \end{bmatrix} = \begin{bmatrix} 0.56 \\ 0.88 \\ 0.88 \\ 0.56 \end{bmatrix}}$$

row

$$\begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \\ u_{42} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.25 & 0 & 0 \\ 0.25 & 0.5 & 0.25 & 0 \\ 0 & 0.25 & 0.5 & 0.25 \\ 0 & 0 & 0.25 & 0.5 \end{bmatrix} \begin{bmatrix} 0.56 \\ 0.88 \\ 0.88 \\ 0.56 \end{bmatrix} + 0.25 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \boxed{\begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \\ u_{42} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.8 \\ 0.8 \\ 0.5 \end{bmatrix}}$$

2- Implicit Method

في هذه الطريقة نعتمد حساب كل صف على الصف الذي يليه وليس الذي يسبقه كما سبق . حيث يكون الحل على الصورة.

$$\boxed{\begin{bmatrix} u_1^{j-1} \\ u_2^{j-1} \\ \vdots \\ u_{n-1}^{j-1} \end{bmatrix} = \begin{bmatrix} (1+2\mu) & -\mu & 0 & \cdots \\ -\mu & (1+2\mu) & -\mu & \cdots \\ 0 & -\mu & (1+2\mu) & -\mu \cdots \\ & & & \end{bmatrix} \begin{bmatrix} u_1^{j} \\ u_2^{j} \\ \vdots \\ u_{n-1}^{j} \end{bmatrix} - \mu \begin{bmatrix} \alpha_0 \\ 0 \\ 0 \\ \vdots \\ \beta_0 \end{bmatrix}}$$

$$\boxed{\alpha_0 = u(0,t) \quad , \quad \beta_0 = u(L,t) \quad , \quad \mu = \frac{\alpha k}{h^2}}$$

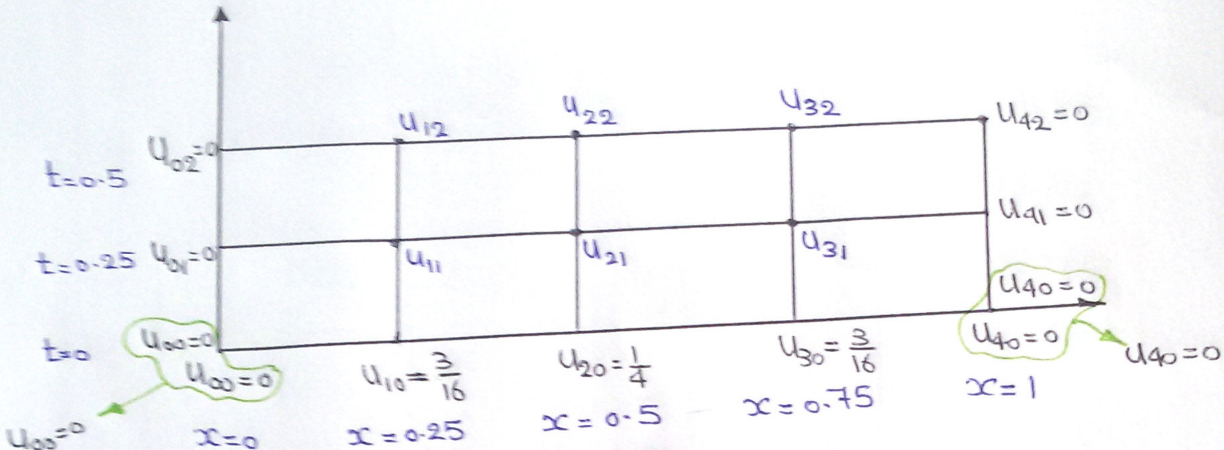
e the implicit method to Solve the heat eq$^{\underline{n}}$.

$u_t = u_{xx}$ $\qquad u(0,t) = u(1,t) = 0 \qquad t \geq 0$

$\qquad\qquad u(x,0) = x(1-x) \qquad 0 \leq x \leq 1$

with $h = k = 0.25$

Sol.



$$\boxed{\mu} = \frac{\alpha k}{h^2} = \boxed{4}$$

- 1^{st} row

$$\begin{bmatrix} \frac{3}{16} \\ \frac{1}{4} \\ \frac{3}{16} \end{bmatrix} = \begin{bmatrix} 9 & -4 & 0 \\ -4 & 9 & 0 \\ 0 & -4 & 9 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} - 4 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} = \begin{bmatrix} 9 & -4 & 0 \\ -4 & 9 & 0 \\ 0 & -4 & 9 \end{bmatrix}^{-1} \begin{bmatrix} \frac{3}{16} \\ \frac{1}{4} \\ \frac{3}{16} \end{bmatrix}$$

* to find inverse

$$A^{-1} = \begin{bmatrix} 9 & -4 & 0 \\ -4 & 9 & 0 \\ 0 & -4 & 9 \end{bmatrix}^{-1} = \frac{1}{|A|} \operatorname{adj}(A) = \frac{1}{441} \operatorname{adj}(A)$$

- Adj(A)

$$\begin{bmatrix} 65 & -36 & 16 \\ -36 & 81 & -36 \\ 16 & -36 & 65 \end{bmatrix} \longrightarrow \begin{bmatrix} 65 & 36 & 16 \\ 36 & 81 & 36 \\ 16 & 36 & 65 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 65 & 36 & 16 \\ 36 & 81 & 36 \\ 16 & 36 & 65 \end{bmatrix} = adj(A)$$

$$\therefore \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} = \frac{1}{441} \begin{bmatrix} 65 & 36 & 16 \\ 36 & 81 & 36 \\ 16 & 36 & 65 \end{bmatrix} \begin{bmatrix} \frac{3}{16} \\ \frac{1}{4} \\ \frac{3}{16} \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} = \begin{bmatrix} 0.0548 \\ 0.0765 \\ 0.0548 \end{bmatrix}}$$

- 2^{nd} row

$$\begin{bmatrix} 0.0548 \\ 0.0765 \\ 0.0548 \end{bmatrix} = \begin{bmatrix} 9 & -4 & 0 \\ -4 & 9 & -4 \\ 0 & -4 & 9 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \end{bmatrix} - 4 \begin{bmatrix} u_{02} \\ 0 \\ u_{42} \end{bmatrix}$$

$$\therefore \begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \end{bmatrix} = \begin{bmatrix} 9 & -4 & 0 \\ -4 & 9 & -4 \\ 0 & -4 & 9 \end{bmatrix}^{-1} \begin{bmatrix} 0.0548 \\ 0.0765 \\ 0.0548 \end{bmatrix}$$

$$\therefore \begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \end{bmatrix} = \frac{1}{441} \begin{bmatrix} 65 & 36 & 16 \\ 36 & 81 & 36 \\ 16 & 36 & 65 \end{bmatrix} \begin{bmatrix} 0.0548 \\ 0.0765 \\ 0.0548 \end{bmatrix} \Rightarrow \boxed{\begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \end{bmatrix} = \begin{bmatrix} 0.0163 \\ 0.023 \\ 0.0163 \end{bmatrix}}$$

ank - Nicolson method

هنا تم الجمع بين أسلوب الـ Explicit والـ Implicit وذلك لتقليل الأخطـ
يكون الحل على الصورة.

$$\begin{bmatrix} (1+\mu) & -\frac{\mu}{2} & 0 & \cdots \\ -\frac{\mu}{2} & (1+\mu) & -\frac{\mu}{2} & 0 \cdots \\ 0 & -\frac{\mu}{2} & (1+\mu) & -\frac{\mu}{2} \cdots \end{bmatrix} \begin{bmatrix} u_1^{j+1} \\ \vdots \\ u_{n-1}^{j+1} \end{bmatrix} = \begin{bmatrix} (1-\mu) & \frac{\mu}{2} & 0 & \cdots \\ \frac{\mu}{2} & (1-\mu) & \frac{\mu}{2} & 0 \cdots \\ 0 & \frac{\mu}{2} & (1-\mu) & \frac{\mu}{2} \cdots \end{bmatrix} \begin{bmatrix} u_1^{j} \\ \vdots \\ u_{n-1}^{j} \end{bmatrix}$$

3- Use Crank-Nicolson method to solve

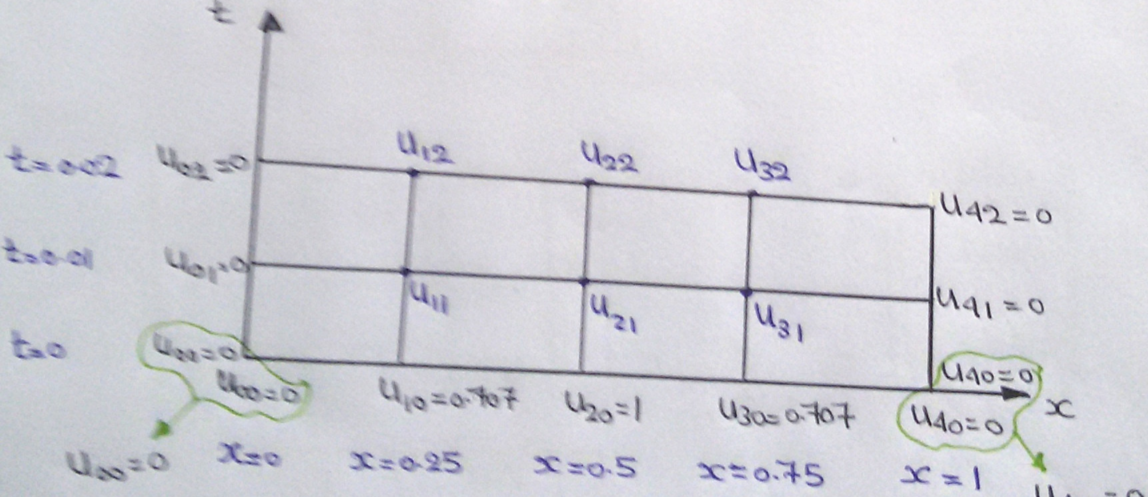
$$25u_{xx} = 4u_t \qquad u(0,t) = u(1,t) = 0$$
$$u(x,0) = \sin \pi x$$

For two levels with $h = 0.25$, $\mu = 1$

Sol.

$$\mu = \frac{\alpha k}{h^2} \Rightarrow k = \frac{\mu h^2}{\alpha}, \quad \alpha = \frac{25}{4}$$

$$\therefore \boxed{k} = \frac{1(0.25)^2}{25/4} = \boxed{0.01}$$



- 1st row

$$\begin{bmatrix} 2 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0.707 \\ 1 \\ 0.707 \end{bmatrix}$$

$$\therefore \begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} = \begin{bmatrix} 2 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0.707 \\ 1 \\ 0.707 \end{bmatrix}$$

$$= \frac{1}{7} \begin{bmatrix} \frac{15}{4} & 1 & \frac{1}{4} \\ 1 & 4 & 1 \\ \frac{1}{4} & 1 & \frac{15}{4} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0.707 \\ 1 \\ 0.707 \end{bmatrix}$$

$$= \begin{bmatrix} 0.071 & 0.29 & 0.071 \\ 0.29 & 0.14 & 0.29 \\ 0.071 & 0.29 & 0.071 \end{bmatrix} \begin{bmatrix} 0.707 \\ 1 \\ 0.707 \end{bmatrix}$$

$$\therefore \boxed{\begin{bmatrix} u_{11} \\ u_{21} \\ u_{31} \end{bmatrix} = \begin{bmatrix} 0.3867 \\ 0.5469 \\ 0.3867 \end{bmatrix}}$$

• 2^{nd} row

$$\begin{bmatrix} 2 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 2 \end{bmatrix} \begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0.3867 \\ 0.5469 \\ 0.3867 \end{bmatrix}$$

$$\therefore \boxed{\begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \end{bmatrix} = \begin{bmatrix} 0.2115 \\ 0.2991 \\ 0.2115 \end{bmatrix}}$$

Hyperbolic eq$^{\underline{n}}$.

$$u_{tt} = \alpha^2 u_{xx} \quad \text{"wave eq}^{\underline{n}}\text{"}$$

$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$

نعتمد كل صف في هذه الطريقة على الصفين السابقين له حيث يتم الحل كما يلي:

- 1st row

$$\boxed{u_i^1 = (1-\mu^2) f_i + \frac{\mu^2}{2}(f_{i+1} + f_{i-1}) + k g_i} \quad i = 1, 2, \cdots, n-1$$

$$\boxed{\mu = \frac{\alpha k}{h}}$$

- 2nd row ↑

$$\begin{bmatrix} u_1^{j+1} \\ u_2^{j+1} \\ \vdots \\ u_{n-1}^{j+1} \end{bmatrix} = \begin{bmatrix} 2(1-\mu^2) & \mu^2 & 0 & \cdots \\ \mu^2 & 2(1-\mu^2) & \mu^2 & \cdots \\ 0 & \mu^2 & 2(1-\mu^2) & \mu^2 \cdots \\ 0 & 0 & \mu^2 & 2(1-\mu^2) \quad \mu^2 \end{bmatrix} \begin{bmatrix} u_1^j \\ u_2^j \\ \vdots \\ u_{n-1}^j \end{bmatrix} - \begin{bmatrix} u_1^{j-1} \\ \vdots \\ u_{n-1}^{j-1} \end{bmatrix}$$

$$\boxed{\mu \leq 1} \longrightarrow \text{"Stable"}$$

4- Approximate the Sol. of wave eq$^{\underline{n}}$.

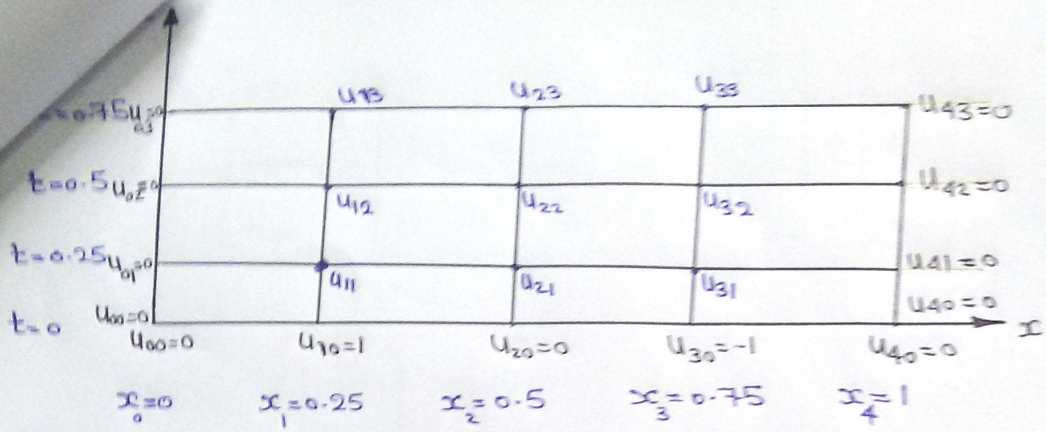
$$u_{tt} = u_{xx} \qquad 0 \leq x \leq 1 \ , \quad 0 \leq t \leq 0.75$$

$$u(0,t) = u(1,t) = 0$$
$$u_t(x,0) = \sin \pi x$$
$$u(x,0) = \sin 2\pi x$$

use $h = k = 0.25$

Sol.

$$\mu = \frac{\alpha k}{h} \Rightarrow \boxed{\mu = 1}$$



- 1st row

$$u_{11} = \frac{1}{2}(\sin 2\pi x_2 + \sin 2\pi x_0) + 0.25 \sin \pi x_1$$

$$\boxed{u_{11} = -0.0569}$$

$$u_{21} = \frac{1}{2}(\sin 2\pi x_3 + \sin 2\pi x_1) + 0.25 \sin \pi x_2$$

$$\boxed{u_{21} = 0.25}$$

$$u_{31} = \frac{1}{2}(\sin 2\pi x_4 + \sin 2\pi x_2) + 0.25 \sin \pi x_3$$

$$\boxed{u_{31} = 0.4105}$$

- 2nd row

$$\begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -0.0569 \\ 0.25 \\ 0.4105 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\therefore \boxed{\begin{bmatrix} u_{12} \\ u_{22} \\ u_{32} \end{bmatrix} = \begin{bmatrix} -0.75 \\ 0.3536 \\ 1.25 \end{bmatrix}}$$

row

$$\begin{bmatrix} u_{13} \\ u_{23} \\ u_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -0.75 \\ 0.3536 \\ 1.25 \end{bmatrix} - \begin{bmatrix} -0.0569 \\ 0.25 \\ 0.4105 \end{bmatrix}$$

$$\therefore \boxed{\begin{bmatrix} u_{13} \\ u_{23} \\ u_{33} \end{bmatrix} = \begin{bmatrix} 0.4105 \\ 0.25 \\ -0.0569 \end{bmatrix}}$$

rse of the matrix

1- matrix 2*2

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longrightarrow A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

ex

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\therefore A^{-1} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

2- matrix 3*3

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & H & k \end{pmatrix} \xrightarrow[\text{المحدد المرافق لكل عنصر}]{\text{نحسب}} \begin{pmatrix} L & M & N \\ O & P & Q \\ R & S & T \end{pmatrix}$$

$$\xrightarrow[\text{الإشارات}]{\text{نطبق}} \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \xrightarrow[\text{المصفوفة}]{\text{ندور}} adj(A)$$

$$\therefore \boxed{A^{-1} = \frac{1}{|A|} adj(A)}$$

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Gauss - Seidal Method

نستخدم هذه الطريقة لإيجاد الحل لمجموعة من المعادلات وهي من طرق الـ "Iterative".

فكرة الحل

• If we have the following eq^{ns}.

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = C_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 = C_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 = C_3$$
$$a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = C_4$$

and $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ as initial.

So

$$x_1 = \frac{C_1 - a_{12}x_2 - a_{13}x_3 - a_{14}x_4}{a_{11}}$$

$$x_2 = \frac{C_2 - a_{21}x_1 - a_{23}x_3 - a_{24}x_4}{a_{22}}$$

$$x_3 = \frac{C_3 - a_{31}x_1 - a_{32}x_2 - a_{34}x_4}{a_{33}}$$

$$x_4 = \frac{C_4 - a_{41}x_1 - a_{42}x_2 - a_{43}x_3}{a_{44}}$$

$$\boxed{X_i = \frac{C_i - \sum_{\substack{j=1 \\ i\neq j}}^{n} a_{ij} X_j}{a_{ii}}}$$

*ملحوظة ← التعويض عن قيمة x يكون بآخر قيمة تم الوصول إليها لـ "X"

$$\boxed{E = \frac{X_{new} - X_{old}}{X_{new}} * 100}$$

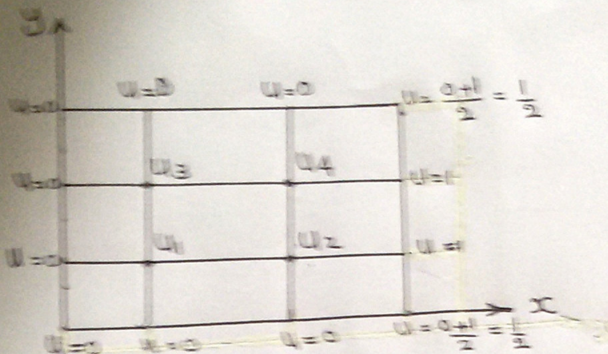
• يتم التكرار حتى نحصل على أقل خطأ أي حتى يحدث ثبات في قيمة "X" (حتى 5 مرات)

-★-

ve the laplace eqn. $\nabla^2 u = 0$ in the region $0 \le x \le 1, 0 \le y \le 1$
der the Boundary Cond.

$u(0,y) = 0$, $u(1,y) = 1$, Let $h = k = \frac{1}{3}$
$u(x,0) = 0$, $u(x,1) = 0$

Sol.



- at u_1

$$-4u_1 + u_2 + u_3 = 0 \quad \text{---} (1)$$

- at u_2

$$u_1 - 4u_2 + u_4 = -1 \quad \text{---} (2)$$

- at u_3

$$u_1 + u_2 - 4u_3 = 0 \quad \text{---} (3)$$

- at u_4

$$u_2 + u_3 - 4u_4 = -1 \quad \text{---} (4)$$

- So

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 1 & -4 & 0 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

- Solving with Jauss-Jordan or Gauss-elimination

$$\left[\begin{array}{cccc|c} -4 & 1 & 1 & 0 & 0 \\ 1 & -4 & 0 & 1 & -1 \\ 1 & 0 & -4 & 1 & 0 \\ 0 & 1 & 1 & -4 & -1 \end{array}\right] \xrightarrow{\frac{R_1}{-4}} \left[\begin{array}{cccc|c} 1 & -0.25 & -0.25 & 0 & 0 \\ 1 & -4 & 0 & 1 & -1 \\ 1 & 0 & -4 & 1 & 0 \\ 0 & 1 & 1 & -4 & -1 \end{array}\right]$$

$$\xrightarrow[-R_1+R_3]{-R_1+R_2} \left[\begin{array}{cccc|c} 1 & -0.25 & -0.25 & 0 & 0 \\ 0 & -3.75 & 0.25 & 1 & -1 \\ 0 & 0.25 & -3.75 & 1 & 0 \\ 0 & 1 & 1 & -4 & -1 \end{array}\right] \xrightarrow{R_2/-3.75}$$

$$\left[\begin{array}{cccc|c} 1 & -0.25 & -0.25 & 0 & 0 \\ 0 & 1 & -0.0667 & -0.267 & 0.267 \\ 0 & 0.25 & -3.75 & 1 & 0 \\ 0 & 1 & 1 & -4 & -1 \end{array}\right] \xrightarrow[-R_2+R_4]{-0.25R_2+R_3}$$

$$\left[\begin{array}{cccc|c} 1 & -0.25 & -0.25 & 0 & 0 \\ 0 & 1 & -0.0667 & -0.267 & 0.267 \\ 0 & 0 & -3.73 & 1.067 & -0.067 \\ 0 & 0 & 1.067 & -3.73 & -1.267 \end{array}\right] \xrightarrow{R_3/-3.73}$$

$$\left[\begin{array}{cccc|c} 1 & -0.25 & -0.25 & 0 & 0 \\ 0 & 1 & -0.0667 & -0.267 & 0.267 \\ 0 & 0 & 1 & -0.286 & 0.018 \\ 0 & 0 & 1.067 & -3.73 & -1.267 \end{array}\right] \xrightarrow{-1.067R_3+R_4}$$

$$\left[\begin{array}{cccc|c} 1 & -0.25 & -0.25 & 0 & 0 \\ 0 & 1 & -0.0667 & -0.267 & 0.267 \\ 0 & 0 & 1 & -0.286 & 0.018 \\ 0 & 0 & 0 & -3.424 & -1.286 \end{array}\right]$$

$\therefore \ -3.424\,u_4 = -1.286 \Rightarrow \boxed{u_4 = 0.3756}$

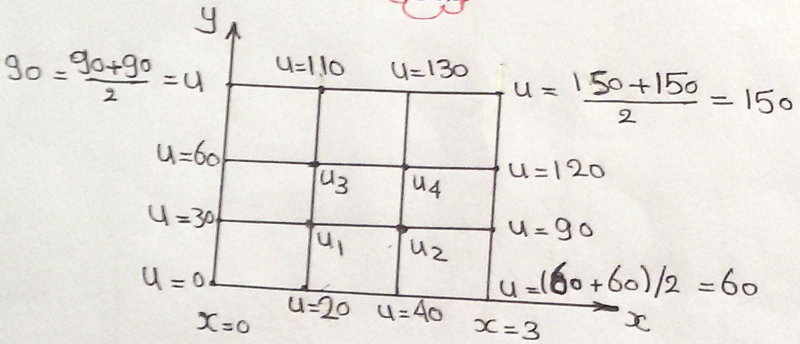
$u_3 = 0.018 + 0.286\,u_4 \Rightarrow \boxed{u_3 = 0.1254}$

$u_2 = 0.267 + 0.0667\,u_3 + 0.267\,u_4 \Rightarrow \boxed{u_2 = 0.3756}$

$u_1 = 0.25\,u_2 + 0.25\,u_3 \Rightarrow \boxed{u_1 = 0.1253}$

2- Solve $\nabla^2 u = 10x$. For $0 \le x \le 3$, $0 \le y \le 3$ and $u(x,y) = 20x + 30y$ on the boundaries take $h = k = 1$

Sol.



$u(0,y) = 30y \quad , \ u(3,y) = 60 + 30y$

$u(x,0) = 20x \quad , \ u(x,3) = 20x + 90$

at u_1

$$-4u_1 + u_2 + u_3 = -50 + 10 = -40 \quad \text{—— (1)}$$

at u_2

$$u_1 - 4u_2 + u_4 = -130 + 20 = -110 \quad \text{—— (2)}$$

at u_3

$$u_1 - 4u_3 + u_4 = -170 + 10 = -160 \quad \text{—— (3)}$$

at u_4

$$u_2 + u_3 - 4u_4 = -250 + 20 = -230 \quad \text{—— (4)}$$

So

$$\begin{bmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} -40 \\ -110 \\ -160 \\ -230 \end{bmatrix}$$

- Solving with Jauss-elimination

$$\left[\begin{array}{cccc|c} -4 & 1 & 1 & 0 & -40 \\ 1 & -4 & 0 & 1 & -110 \\ 1 & 0 & -4 & 1 & -160 \\ 0 & 1 & 1 & -4 & -230 \end{array}\right] \xrightarrow{R_1/4} \left[\begin{array}{cccc|c} 1 & -0.25 & -0.25 & 0 & 10 \\ 1 & -4 & 0 & 1 & -110 \\ 1 & 0 & -4 & 1 & -160 \\ 0 & 1 & 1 & -4 & -230 \end{array}\right]$$

$$\xrightarrow[-R_1+R_2]{-R_1+R_3} \left[\begin{array}{cccc|c} 1 & -0.25 & -0.25 & 0 & 10 \\ 0 & -3.75 & 0.25 & 1 & -120 \\ 0 & 0.25 & -3.75 & 1 & -170 \\ 0 & 1 & 1 & -4 & -230 \end{array}\right] \xrightarrow{R_2/-3.75}$$

$$\left[\begin{array}{cccc|c} 1 & -0.25 & -0.25 & 0 & 10 \\ 0 & 1 & -0.067 & -0.267 & 32 \\ 0 & 0.25 & -3.75 & 1 & -170 \\ 0 & 1 & 1 & -4 & -230 \end{array}\right] \xrightarrow[-R_2+R_4]{0.25R_2+R_1,\ -0.25R_2+R_3}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -0.267 & -0.067 & 18 \\ 0 & 1 & -0.067 & -0.267 & 32 \\ 0 & 0 & -3.73 & 1.067 & -178 \\ 0 & 0 & 1.067 & -3.733 & -262 \end{array}\right] \xrightarrow{R_3/-3.73}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -0.267 & -0.067 & 18 \\ 0 & 1 & -0.067 & -0.267 & 32 \\ 0 & 0 & 1 & -0.286 & 47.7 \\ 0 & 0 & 1.067 & -3.733 & -262 \end{array}\right] \xrightarrow[0.067R_3+R_2,\ -1.067R_3+R_4]{0.267R_3+R_1}$$

$$\left[\begin{array}{cccc|c} & 0 & 0 & -0.143 & 30.74 \\ 0 & 1 & 0 & -0.286 & 35.2 \\ 0 & 0 & 1 & -0.286 & -47.7 \\ 0 & 0 & 0 & -3.43 & -312.9 \end{array}\right] \xrightarrow{R_4 / -3.43}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -0.143 & 30.74 \\ 0 & 1 & 0 & -0.286 & 35.2 \\ 0 & 0 & 1 & -0.286 & -47.7 \\ 0 & 0 & 0 & 1 & 91.22 \end{array}\right] \xrightarrow[0.286 R_4 + R_3]{0.143 R_4 + R_1 \quad 0.286 R_4 + R_2}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 43.78 \\ 0 & 1 & 0 & 0 & 61.3 \\ 0 & 0 & 1 & 0 & 73.8 \\ 0 & 0 & 0 & 1 & 91.22 \end{array}\right] \Rightarrow \boxed{\begin{array}{l} u_1 = 43.78 \\ u_2 = 61.3 \\ u_3 = 73.8 \\ u_4 = 91.22 \end{array}}$$

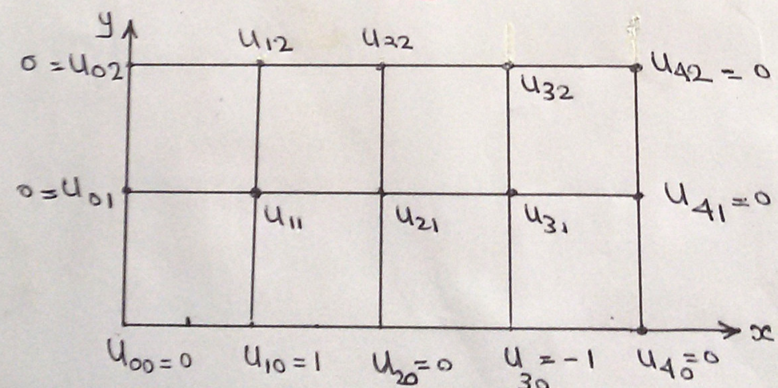
3- Approximate the Solution of the wave eqn.

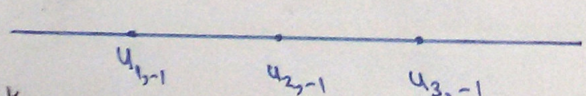
$u_{tt} = u_{xx}$, $0 \le x \le 1$, $0 \le t \le 0.5$

$u(0,t) = u(1,t) = 0$, $u_t(x,0) = \sin \pi x$, $u(x,0) = \sin(2\pi x)$

use $h = 0.25$, $k = 0.25$

Sol.





$$\lambda = \frac{\alpha k}{h} = 1$$

- at u_{11}

$\therefore \; u_{11} = u_{00} + u_{20} - u_{1,-1}$